

Complex spherical codes with two inner products

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Abstract

A finite set X in a complex sphere is called a complex spherical 2-code if the number of inner products between two distinct vectors in X is equal to 2. In this paper, we characterize the tight complex spherical 2-codes by doubly regular tournaments or skew Hadamard matrices. We also give certain maximal 2-codes relating to skew-symmetric D -optimal designs. To prove them, we show the smallest embedding dimension of a tournament into a complex sphere by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix.

Key words: complex spherical s -code, doubly regular tournament, skew Hadamard matrix, skew-symmetric D -optimal design, representable graph, main angle, main eigenvalue, graph spectrum.

1 Introduction

Let X be a finite set of points on the complex unit sphere $\Omega(d)$ in \mathbb{C}^d . The *angle set* $A(X)$ is defined to be

$$A(X) = \{x^*y \mid x, y \in X, x \neq y\},$$

where x^* is the transpose conjugate of a column vector x . A finite set X is called a *complex spherical s -code* if $|A(X)| = s$ and $A(X)$ contains an imaginary number. The value s is called the *degree* of X . For $X, X' \subset \Omega(d)$, we say that X is *isomorphic* to X' if there exists a unitary transformation from X to X' . An s -code $X \subset \Omega(d)$ is said to be *largest* if X has the largest possible cardinality in all s -codes in $\Omega(d)$. One of major problems on s -codes is to classify largest s -codes for given s and d .

We will survey Euclidean finite sets with only s distances. For $X \subset \mathbb{R}^d$, we define

$$D(X) = \{d(x, y) \mid x, y \in X, x \neq y\},$$

where $d(x, y)$ is the Euclidean distance of x and y . A finite set X is called an *s -distance set* if $|D(X)| = s$ holds. We have an upper bound for the size of an s -distance set in \mathbb{R}^d , namely

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$|X| \leq \binom{d+s}{s}$ [2]. Clearly the largest 1-distance set in \mathbb{R}^d is the regular simplex for any d . Largest 2-distance sets in \mathbb{R}^d are classified for $d \leq 7$ [9, 11]. Largest s -distance sets in \mathbb{R}^2 are classified for $s \leq 5$ [10, 19, 20]. The largest 3-distance set in \mathbb{R}^3 is the vertex set of the icosahedron [21]. The classification of largest s -distance sets is still open for others (s, d) . A largest 2-distance set in \mathbb{R}^8 is given in [11], and it attains the upper bound.

A spherical s -distance set particularly deserves attention because of the connection to association schemes or spherical t -designs (see [7, 1] for details). A subset X of S^{d-1} is called a *spherical t -design* if for any polynomial f in d variables of degree at most t , the following equality holds:

$$\frac{1}{|S^{d-1}|} \int_{S^{d-1}} f(x) dx = \frac{1}{|X|} \sum_{x \in X} f(x),$$

where $|S^{d-1}|$ is the volume of S^{d-1} . If a spherical t -design X of degree s satisfies $t \geq 2s - 2$, then X has the structure of a Q -polynomial association scheme [7]. The size of an s -distance set in S^{d-1} is smaller than or equal to $\binom{d+s-1}{s} + \binom{d+s-2}{s-1}$ [7]. An s -distance set X is said to be *tight* if X attains this bound. A tight s -distance set becomes a minimal spherical t -design and satisfies $t = 2s$ [7]. The classification of tight s -distance sets is one of the most interesting problems, and this has been solved except for $s = 2$ [4]. A largest 2-distance set on S^{d-1} is determined for $d \leq 93$ ($d \neq 46, 78$) [13, 5]. A largest 3-distance set on S^{d-1} is determined for $d = 2, 3, 8, 22$ [21, 14].

A simple graph $G = (V, E)$ is *representable* in \mathbb{R}^d if there is an embedding $\sigma : V \rightarrow \mathbb{R}^d$ such that

$$d(\sigma(a), \sigma(b)) = \begin{cases} \alpha & \text{if } (a, b) \in E, \\ \beta & \text{otherwise,} \end{cases}$$

for some $\alpha, \beta \in \mathbb{R}$. For a simple graph G , Roy [18] gave an explicit expression of the minimal dimension d such that G is representable in \mathbb{R}^d in terms of the multiplicity of the smallest or second-smallest eigenvalue of A . This embedding of a graph is useful for the classification of 2-distance sets [9, 11].

Roy and Suda [17] gave the complex analogue of the spherical s -distance set theory. Complex spherical s -codes are closely related to complex spherical designs or non-symmetric association schemes. In this paper, we consider a complex spherical 2-code $X \subset \Omega(d)$. If X satisfies $A(X) \subset \mathbb{R}$, then the Gram matrix of X is real, and X can be embedded into \mathbb{R}^d . We may assume $A(X)$ contains an imaginary number α , and $A(X) = \{\alpha, \bar{\alpha}\}$, where $\bar{\alpha}$ is the conjugate of α . We have a natural upper bound [17]:

$$|X| \leq \begin{cases} 2d + 1 & \text{if } d \text{ is odd,} \\ 2d & \text{if } d \text{ is even.} \end{cases} \quad (1.1)$$

A 2-code X is said to be *tight* if X attains the bound (1.1). This is known as the *absolute bound*.

A *tournament* is a directed graph obtained by assigning a direction for each edge in an undirected complete graph. Formally, a tournament is a pair (V, E) such that the vertex set V is a finite set and the edge set $E \subset V \times V$ satisfies $E \cap E^T = \emptyset$ and $E \cup E^T \cup \{(x, x) \mid x \in V\} = V \times V$, where $E^T := \{(x, y) \mid (y, x) \in E\}$. A complex spherical 2-code X has the structure of a tournament (X, E) , where $E = \{(x, y) \in X \times X \mid x^*y = \alpha\}$. A tournament (V, E) is *representable in $\Omega(d)$* if there exists a mapping φ from V to $\Omega(d)$ such that for all

distinct $x, y \in V$,

$$\varphi(x)^* \varphi(y) = \begin{cases} \alpha & \text{if } (x, y) \in E, \\ \bar{\alpha} & \text{if } (y, x) \in E, \end{cases}$$

where α is an imaginary number with $\text{Im}(\alpha) > 0$. Such a mapping φ is said to be a *representation* of a tournament. We identify a representation with the image of the representation. Two tournaments $G = (V, E), G' = (V', E')$ are *isomorphic* if there is a bijection from V to V' such that $(x, y) \in E$ if and only if $(f(x), f(y)) \in E'$. For two tournaments G and G' , if G is not isomorphic to G' , then a representation of G is not isomorphic to that of G' . Let $\text{Rep}(G)$ denote the smallest d such that G is representable in $\Omega(d)$. The *Seidel matrix* of G is defined to be $\sqrt{-1}(A - A^T)$, where A is the adjacency matrix of G . In Section 3, we determine $\text{Rep}(G)$ by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix of G .

A tournament G is said to be *doubly regular* if the number of the neighbors of a vertex does not depend on the choice of the vertex and the number of the common neighbors of a pair of distinct vertices does not depend on the choice of the pair. An $n \times n$ (± 1) -matrix of H is called a *skew Hadamard matrix* if $H + H^T = 2I$ and $HH^T = nI$, where I is the identity matrix. Let $X \subset \Omega(d)$ be a 2-code, and A the adjacency matrix of the tournament obtained from X . It is known that the existence of a doubly regular tournament of $4d + 3$ vertices is equivalent to that of a skew Hadamard matrix of order $4d + 4$ [16]. In Section 4, we give the following characterizations of tight 2-codes and 2-codes with $n = 2d$ where d is odd.

- (1) For odd d , X is a tight complex 2-code if and only if A is the adjacency matrix of a doubly regular tournament.
- (2) For even d , X is a tight complex 2-code if and only if $I + A - A^T$ is a skew Hadamard matrix.
- (3) For odd d , X is a complex 2-code with $n = 2d$ if and only if either A is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex, or its Seidel matrix S satisfies that S^2 is permutationally similar to

$$\begin{pmatrix} kI + lJ & 0 \\ 0 & kI + lJ \end{pmatrix},$$

for some positive integers k, l .

We note that the last case in (3) includes skew-symmetric D -optimal designs [8, 23]. The table of the number of non-isomorphic tight 2-codes in $\Omega(d)$ for $d \leq 14$ is obtained by a computer calculation based on Theorem 3.2 in [3].

2 Results on main eigenvalues

In this section we give results on main eigenvalues of a Hermitian matrix which will be used later. Let H be a Hermitian matrix of size n with s distinct eigenvalues $\tau_1 < \dots < \tau_s$. Let E_i be the orthogonal projection matrix onto the eigenspace corresponding to τ_i . The *main angle* β_i of τ_i is defined to be the value

$$\beta_i = \frac{1}{\sqrt{n}} \sqrt{(E_i \cdot j)^* (E_i \cdot j)},$$

where j is the all-ones vector. It is clear that $0 \leq \beta_i \leq 1$ and $\sum_{i=1}^s \beta_i^2 = 1$.

Let J denote the all-ones matrix.

Lemma 2.1 ([15]). *Let H be a Hermitian matrix of size n with s distinct eigenvalues $\tau_1 < \dots < \tau_s$. Let β_i be the main angle of τ_i . Let $M = H + aJ$, where a is a complex number. Then*

$$P_M(x) = P_H(x) \left(1 + a \sum_{i=1}^s \frac{n\beta_i^2}{\tau_i - x} \right),$$

where P_M is the characteristic polynomial of matrix M .

An eigenvalue τ_i is said to be *main* if $\beta_i \neq 0$.

Theorem 2.2. *Let H be a Hermitian matrix of size n , and $M = H + aJ$, where a is a real number. Let $\tau_1 < \tau_2 < \dots < \tau_r$ be the distinct main eigenvalues of H , and β_i the main angle of τ_i . Let $\mu_1 < \mu_2 < \dots < \mu_s$ be the distinct main eigenvalues of M . Then $r = s$ holds, and*

$$f(x) = \prod_{i=1}^r (\mu_i - x) = \prod_{i=1}^r (\tau_i - x) \left(1 + a \sum_{j=1}^r \frac{n\beta_j^2}{\tau_j - x} \right). \quad (2.1)$$

Moreover, if $a > 0$, then $\tau_1 < \mu_1 < \tau_2 < \dots < \tau_r < \mu_r$, and if $a < 0$, then $\mu_1 < \tau_1 < \mu_2 < \dots < \mu_r < \tau_r$.

Proof. By Lemma 2.1, we have the equality

$$\prod_{i=1}^s (\mu_i - x) = \prod_{i=1}^r (\tau_i - x) \left(1 + a \sum_{j=1}^r \frac{n\beta_j^2}{\tau_j - x} \right). \quad (2.2)$$

By comparing the degrees of the polynomials in both sides, we obtain $s = r$.

Let $f(x)$ be the polynomial in (2.2). It is easily shown that for $a > 0$,

$$\begin{aligned} f(\tau_i) &> 0, \text{ if } i \equiv 1 \pmod{2}, \\ f(\tau_i) &< 0, \text{ if } i \equiv 0 \pmod{2}, \\ \lim_{x \rightarrow \infty} f(x) &< 0, \text{ if } r \equiv 1 \pmod{2}, \\ \lim_{x \rightarrow \infty} f(x) &> 0, \text{ if } r \equiv 0 \pmod{2}. \end{aligned}$$

This implies that $\tau_1 < \mu_1 < \tau_2 < \dots < \tau_r < \mu_r$. By the same manner for $H = M - aJ$ with $a < 0$, we can show $\mu_1 < \tau_1 < \mu_2 < \dots < \mu_r < \tau_r$. \square

3 Representations of a tournament

In this section, we determine $\text{Rep}(G)$ by the multiplicity of the smallest or second-smallest eigenvalue of the Seidel matrix of G . Let $G = (V, E)$ be a tournament with n vertices. The *adjacency matrix* A of G is the matrix indexed by the vertex set V , with entries given by

$$A_{xy} = \begin{cases} 1 & \text{if } (x, y) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

The Gram matrix of a representation of G , with adjacency matrix A , can be expressed by

$$\alpha A + \overline{\alpha} A^T - \tau I,$$

where α is an imaginary number, and τ is a negative real number. Note that τ should be the smallest eigenvalue of $\alpha A + \overline{\alpha} A^T$ to minimize the rank. To determine $\text{Rep}(G)$, we will consider α for which the multiplicity of the smallest eigenvalue of $\alpha A + \overline{\alpha} A^T$ is maximum.

Theorem 3.1. *Let G be a tournament with n vertices, and A the adjacency matrix. Let $\tau_1 < \tau_2 < \dots < \tau_s$ be the distinct eigenvalues of $S = \sqrt{-1}(A - A^T)$, β_i the main angle of τ_i , and m_i the multiplicity of τ_i . Let α be the angle with $\text{Im}(\alpha) > 0$ of the representation of G in $\Omega(\text{Rep}(G))$. Then the following hold.*

- (1) *If $\beta_1 = 0$, then $\text{Rep}(G) = n - m_1 - 1$, and $\alpha = (1 - c_1\sqrt{-1})/(1 + c_1\tau_1)$, where $c_1 = \sum_{i=2}^s n\beta_i^2/(\tau_i - \tau_1)$.*
- (2) *If $\beta_1 \neq 0$, and $m_1 > 1$, then $\text{Rep}(G) = n - m_1$, and $\alpha = -\sqrt{-1}/\tau_1$.*
- (3) *If $m_1 = 1$, $\beta_2 = 0$, and $c_2 < 0$, then $\text{Rep}(G) = n - m_2 - 1$, and $\alpha = (1 - c_2\sqrt{-1})/(1 + c_2\tau_2)$, where $c_2 = n\beta_1^2/(\tau_1 - \tau_2) + \sum_{i=3}^s n\beta_i^2/(\tau_i - \tau_2)$.*
- (4) *Otherwise $\text{Rep}(G) = n - 1$.*

Proof. For $\alpha' = a + \sqrt{-1}$ with $a \in \mathbb{R}$, we have

$$\alpha' A + \overline{\alpha'} A^T = aJ + \sqrt{-1}(A - A^T) - aI.$$

The multiplicity of the smallest eigenvalue of $\alpha' A + \overline{\alpha'} A^T$ is equal to that of $M = aJ + \sqrt{-1}(A - A^T)$. We would like to find $a \in \mathbb{R}$ such that the multiplicity of the smallest eigenvalue of M is maximum. Let $\tau_{k_1} < \dots < \tau_{k_r}$ be the distinct main eigenvalues of S , and $\mu_{l_1} < \dots < \mu_{l_r}$ those of M . Let $f(x)$ be the polynomial defined as in Theorem 2.2.

(1) By $\beta_1 = 0$, we have $\tau_1 < \tau_{k_1}$. We would like to find $a \in \mathbb{R}$ such that $\mu_{l_1} = \tau_1$. For such a , the multiplicity of the smallest eigenvalue τ_1 of M is maximum, and equal to $m_1 + 1$. By Theorem 2.2, $\mu_{l_1} = \tau_1$ if and only if $f(\tau_1) = 0$, namely, $a = -1/c_1$. Therefore $\text{Rep}(G) = n - m_1 - 1$ for $a = -1/c_1$. By rescaling the diagonal entries of $\alpha' A + \overline{\alpha'} A^T - (\tau_1 - a)I$ to 1, we obtain $\alpha = (1 - c_1\sqrt{-1})/(1 + c_1\tau_1)$.

(2) Since $\beta_1 \neq 0$, we have $\tau_1 = \tau_{k_1} \neq \mu_{l_1}$ by Theorem 2.2. Therefore, if $a \neq 0$, the multiplicity of the smallest eigenvalue of M is at most $m_1 - 1$. Thus, for $a = 0$, the multiplicity of the smallest eigenvalue of M is maximum, and equal to m_1 . Hence $\text{Rep}(G) = n - m_1$, and $\alpha = -\sqrt{-1}/\tau_1$.

(3) By $c_2 < 0$, we have $\beta_1 > 0$ and τ_1 is a main eigenvalue. We would like to find $a \in \mathbb{R}$ such that $\mu_{l_1} = \tau_2$. For such a , the multiplicity of the smallest eigenvalue τ_2 of M is maximal, and it is $m_2 + 1$. By Theorem 2.2, $\mu_{l_1} = \tau_2$ if and only if $f(\tau_2) = 0$ and $a > 0$, namely, $a = -1/c_2$ and $c_2 < 0$. Therefore we obtain $\text{Rep}(G) = n - m_2 - 1$, and $\alpha = (1 - c_2\sqrt{-1})/(1 + c_2\tau_2)$.

(4) If $a = 0$ and $m_1 = 1$, then the multiplicity of the smallest eigenvalue of M is clearly 1.

Suppose $\beta_1 \neq 0$, $m_1 = 1$, $\beta_2 = 0$, and $c_2 \geq 0$. If $a > 0$ holds, then $\mu_{l_1} < \tau_2$ by $f(\tau_2) < 0$ and $\lim_{x \rightarrow -\infty} f(x) > 0$. If $a < 0$ holds, then $\mu_{l_1} < \tau_1$ by Theorem 2.2. The multiplicity of the smallest eigenvalue μ_{l_1} of M is 1.

Suppose $\beta_1 \neq 0$, $m_1 = 1$, $\beta_2 \neq 0$. Then for any $a \neq 0$, the multiplicity of the smallest eigenvalue μ_{l_1} of M is 1 by Theorem 2.2.

From the above facts, $\text{Rep}(G) = n - 1$ follows. \square

Note that the conditions (1)–(4) in Theorem 3.1 are disjoint. A tournament which satisfies the condition (i) in Theorem 3.1 is said to be of *Type (i)* for $i = 1, \dots, 4$. There is a tournament of each type. Lemmas 4.3, 4.4, and Remark 4.9 give examples of Type (1), (2), and (3), respectively.

4 Tight complex spherical 2-codes

In this section, we give bounds on complex spherical 2-codes. We also characterize the tight 2-codes and 2-codes in $\Omega(d)$ with $n = 2d$ vertices, where d is odd in terms of doubly regular tournaments, skew Hadamard matrix and some skew symmetric $(0, \pm 1)$ -matrices including skew-symmetric D -optimal designs as an application of Theorem 3.1.

Let X be a finite subset in $\Omega(d)$ of size n with degree 2, and let A be the adjacency matrix of X . Example 6.3 in [17] shows that the following are equivalent:

- (1) $|X| = 2d + 1$.
- (2) $\{I, A, J - A - I\}$ forms the set of adjacency matrices of a non-symmetric association scheme of class 2.

Theorem 4.1. *Let X be a finite subset in $\Omega(d)$ of size n with degree 2, and let A be the adjacency matrix of X . If d is odd, $|X| \leq 2d + 1$ holds. Equality holds if and only if A is the adjacency matrix of a doubly regular tournament.*

Proof. The absolute bound (1.1) shows that $|X| \leq 2d + 1$ holds. Example 6.3 in [17] shows that equality holds if and only if $\{I, A, J - A - I\}$ forms the set of adjacency matrices of a non-symmetric association scheme of class 2. The latter condition is equivalent to the condition that A is the adjacency matrix of a doubly regular tournament. \square

To prove Theorems 4.7, 4.8, we need the following lemmas.

Lemma 4.2. *There exists no tournament A of Type (1) with $n = 2d$ vertices and the spectrum $\{(-\theta)^{d-1}, 0^2, (\theta)^{d-1}\}$ where $0 < \theta$.*

Proof. Suppose that there exists such a tournament with Seidel matrix S . It holds that $Sj = 0$ because $\beta_1 = \beta_3 = 0$ and the remaining eigenvalues are all 0. However it does not happen because $n = 2d$. \square

Lemma 4.3. *Let d be an integer at least 3. Let A be the adjacency matrix of a tournament of Type (1) with $n = 2d$ vertices and the spectrum $\{(-\theta)^{d-1}, (-\phi)^1, (\phi)^1, (\theta)^{d-1}\}$ where $0 < \phi < \theta$. Then d is odd and A is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex.*

Proof. Since the entries of S^2 are integers, the eigenvalues of S^2 are algebraic integers. Therefore θ^2 and ϕ^2 are integer because their multiplicities $2d - 2$ and 2 are different. From taking the trace of S^2 , it follows that the possibility of (θ^2, ϕ^2) is $(2d + 1, 1)$ or $(2d, d)$.

For the first case, A is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex [15, Theorem 1.1]. Thus $n + 1 = 2d + 1$ must be congruent to 3 modulo 4, which implies that d is odd.

For the second case, consider $\theta^2 I - S^2$. Since $\theta^2 I - S^2$ is positive semidefinite and the diagonal entries are all 1, the absolute value of an off-diagonal entry of this matrix must be at

most 1. In fact they must be zero because the size of the matrix $\theta^2 I - S^2$ is even. Therefore $S^2 = (\theta^2 - 1)I$, which contradicts the fact that S^2 has the other eigenvalue ϕ^2 . \square

Lemma 4.4. *Let A be the adjacency matrix of a tournament of Type (2) with $n = 2d$ vertices and the spectrum $\{(-\theta)^d, (\theta)^d\}$ where $0 < \theta$. Then d is even and $I + A - A^T$ is a skew Hadamard matrix.*

Proof. The fact that $I + A - A^T$ is a skew Hadamard matrix follows from direct calculation, and thus d must be even. \square

Lemma 4.5. *Let A be the adjacency matrix of a tournament of Type (3) with the spectrum $\{(-\theta)^1, (-\phi)^{d-1}, (\phi)^{d-1}, (\theta)^1\}$ where $0 < \phi < \theta$. Then d is odd and the Seidel matrix S satisfies that S^2 is permutationally similar to*

$$\begin{pmatrix} kI + lJ & 0 \\ 0 & kI + lJ \end{pmatrix}, \quad (4.1)$$

for some positive integers k, l .

Proof. By the condition of Type (3), $\beta_2 = \beta_3 = 0$ and $\beta_1 = \beta_4 = 1/\sqrt{2}$ hold. Consider the eigenspaces of $S^2 - \phi^2 I$. The main angle condition of S implies that the all-ones vector is an eigenvector of $S^2 - \phi^2 I$ corresponding to the eigenvalue $\theta^2 - \phi^2$. Since the multiplicity of $\theta^2 - \phi^2$ is two, let x be the remaining normalized real eigenvector orthogonal to j . Then it holds that

$$S^2 = \phi^2 I + (\theta^2 - \phi^2)((1/n)J + xx^T).$$

Comparing the diagonal entries, we observe that $n - 1 = \phi^2 + (\theta^2 - \phi^2)(1/n + x_i^2)$ for each i , where x_i is the i -th entry of x . This implies that x_i^2 is independent of the choice of i . Since the vector x is normalized, we obtain $x_i = \pm 1/\sqrt{n}$. The assumption that x is orthogonal to the all-ones vector shows that each $\pm 1/\sqrt{n}$ appears in the entries of x exactly same times. After some permutation of entries, we may assume that the first half entries of x are $1/\sqrt{n}$ which means S^2 has the form

$$S^2 = \begin{pmatrix} \phi^2 I + \frac{2(\theta^2 - \phi^2)}{n} J & 0 \\ 0 & \phi^2 I + \frac{2(\theta^2 - \phi^2)}{n} J \end{pmatrix}.$$

Since a vector $S(j + \sqrt{n}x)$ is written as a linear combination of j, x and $S = \sqrt{-1}(2A - J + I)$, we have

$$A \begin{pmatrix} j \\ 0 \end{pmatrix} = \begin{pmatrix} aj \\ bj \end{pmatrix}$$

for some a, b . Letting A_1 be the principal submatrix of A lying the first d rows and columns, then $A_1 j = aj$, namely A_1 is the adjacency matrix of a regular tournament of order d . This implies d must be odd. \square

Lemma 4.6. *Let X be a finite subset in $\Omega(d)$ with degree 2 and size $n = 2d$. The possibilities of the spectrum of $S = \sqrt{-1}(A - A^T)$ are as follows:*

- (i) X is of Type (1) with the spectrum $\{(-\theta)^{d-1}, 0^2, (\theta)^{d-1}\}$.

- (ii) X is of Type (1) with the spectrum $\{(-\theta)^{d-1}, (-\phi)^1, (\phi)^1, (\theta)^{d-1}\}$ with $0 < \phi < \theta$.
- (iii) X is of Type (2) with the spectrum $\{(-\theta)^d, (\theta)^d\}$.
- (iv) X is of Type (3) with the spectrum $\{(-\theta)^1, (-\phi)^{d-1}, (\phi)^{d-1}, (\theta)^1\}$ with $0 < \phi < \theta$.

Proof. Follows from Theorem 3.1. \square

Theorem 4.7. *Let X be a finite subset of $\Omega(d)$ of size n with degree 2, and let A be the adjacency matrix of X . If d is even, $|X| \leq 2d$ holds. Equality holds if and only if $I + A - A^T$ is a skew Hadamard matrix.*

Proof. A necessary condition for the existence of doubly regular tournaments is $|X| \equiv 3 \pmod{4}$, namely d is odd. Therefore if d is even then $|X| < 2d + 1$, that is, $|X| \leq 2d$ holds.

Let H be a skew Hadamard matrix of size n . Then n must be a multiple of 4. Define $S = \sqrt{-1}(H - I)$ and $A = \frac{1}{2}(-\sqrt{-1}S + J - I)$. Then the spectrum of S is $\{(-\sqrt{n-1})^{n/2}, (\sqrt{n-1})^{n/2}\}$. Thus A is of Type (2) and the minimum embedding dimension is $d = n/2$. Therefore $n = 2d$.

Let X be a finite subset of $\Omega(d)$ with degree 2 and size $n = 2d$. First we consider the case $d = 2$. In this case, the classification of tournaments of order 4 is given [12] and the list of A are

$$\begin{aligned}
 \text{(a)} \quad & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 3, & \text{(b)} \quad & \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 2, \\
 \text{(c)} \quad & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 3, & \text{(d)} \quad & \begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ with } \text{Rep}(G) = 2.
 \end{aligned}$$

The tournaments (b) and (d) satisfy $n = 2d$, and in these cases, $I + A - A^T$ is a skew Hadamard matrix.

Next we consider the case where $d \geq 4$. By Lemmas 4.2–4.6 and the assumption that d is even, $I + A - A^T$ is a skew Hadamard matrix as desired. \square

Theorem 4.8. *Let d be an odd integer at least 3. Let X be a finite subset of $\Omega(d)$ of size n with degree 2, and let A be the adjacency matrix of the tournament obtained from X . The finite subset X has the size $n = 2d$ if and only if one of the following occurs:*

- (i) A is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex.
- (ii) the Seidel matrix S satisfies that S^2 is permutationally similar to

$$\begin{pmatrix} kI + lJ & 0 \\ 0 & kI + lJ \end{pmatrix}, \tag{4.2}$$

for some positive integers k, l .

Proof. Let A be the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex. From Theorem 1.1 and Remark 2.8 in [15] A is of Type (1) and the minimum embedding dimension is $d = n/2$. Therefore $n = 2d$.

Let S be the Seidel matrix which satisfies (4.2). By the block form of S^2 , the eigenvalues S^2 are $k + ld, k$ with multiplicities $2, 2d - 2$ respectively. Thus the eigenvalues of S are $\pm\sqrt{k + ld}, \pm\sqrt{k}$ with multiplicities $1, d - 1$ respectively. The eigenvectors of S^2 corresponding to $k + ld$ are the all-ones vector and the (± 1) -vector with the first d entries equal to 1 and the last d entries equal to -1 . This implies that main angles of S corresponding to $\pm\sqrt{k}$ are 0. Thus the adjacency matrix of S is of Type (3) and the minimum embedding dimension $d = n/2$. Therefore $n = 2d$.

Let X be a finite subset in $\Omega(d)$ with degree 2 and size $n = 2d$. By Lemmas 4.2–4.6 and the assumption that d is odd, either A is the adjacency matrix of an induced subgraph of a doubly regular tournament by deleting a vertex or the Seidel matrix S satisfies that S^2 is permutationally similar to (4.2) as desired. \square

Remark 4.9. Chadjiapantelis and Kounias [6, Theorem] showed that supplementary difference sets construct (± 1) -matrix S satisfying (4.2).

For the Seidel matrix S satisfying (4.2) with $(k, l) = (n - 3, 2)$, $\sqrt{-1}S + I$ is known as the D -optimal designs [8, 23]. Let A_1, A_2 be the adjacency matrices of doubly regular tournaments of same order. Then a tournament of the adjacency matrix

$$\begin{pmatrix} A_1 & J \\ 0 & A_2 \end{pmatrix}$$

satisfies (4.2) for $(k, l) = (d, d - 1)$. For $d = 2$, this example corresponds to a skew D -optimal design.

When d is odd, the number of tight 2-codes in $\Omega(d)$ is equal to that of doubly regular tournaments of order $2d + 1$. When d is even, the number of tight 2-codes in $\Omega(d)$ is that of tournaments in the switching classe of the tournament obtained by adding one vertex with no outward edges and all possible inward edges to a doubly regular tournament. If we use a computer, the number of non-isomorphic tournaments in a switching class can be calculated by Theorem 3.2 in [3]. Therefore if doubly regular tournaments are classified, then we can determine the number of tight 2-codes. Doubly regular tournaments have been classified for order at most 27 [22], and we can find the catalogue in [12]. Note that non-isomorphic doubly regular tournaments may be in the same switching class. By using a computer calculation based on Theorem 3.2 in [3], we can give the number of tight 2-codes as Table 1.

d	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$ X $	3	4	7	8	11	12	15	16	19	20	23	24	27	28
$\#$	1	2	1	4	1	8	2	240	2	8956	37	11339044	722	9897616700

Table 1: Tight complex 2-code X in $\Omega(d)$

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